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Solutions to the Yang–Baxter equation for the spinor representations of $q-B_l$

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Abstract. In this paper, both trigonometric and rational solutions to the Yang-Baxter equation associated with the spinor representations of the quantum B_l universal enveloping algebras are obtained. The corresponding representations of the braid group and the link polynomials are also computed through a standard method. The quantum Clebsch–Gordan matrix, the quantum projectors and the solutions associated with the spinor representation of the quantum B_3 are presented explicitly.

1. Introduction

The solution of the Yang–Baxter equation [1] plays a key role in some physical and mathematical fields, such as the construction of solvable statistical models [2], the computation of the representations of the braid group and link polynomials [3], the evaluation of the correlation functions in the conformal field theories [4], and so on.

The trigonometric solutions associated with some representations of the quantum Lie universal enveloping algebras $q-\mathscr{L}$ were obtained. The solutions for the minimal representations of $q-A_l$ were obtained earlier [5]. Ogievetsky and Wiegmann [6] listed the solutions for the minimal representations of some $q-\mathscr{L}$. Jimbo [7] presented a systematic method for constructing the spectrum-dependent solutions, and computed [8] the solutions for the minimal representations of $q-B_l$, $q-C_l$ and $q-D_l$. In terms of this method, Kuniba [9] computed the solutions for the minimal representation of $q-G_2$, and we [10] computed the solutions for the minimal representations of $q-F_4$, $q-E_6$ and $q-E_7$. This method is also effective for constructing the solutions beyond the minimal representations, especially for $q-A_l$. We [11] computed the solution for the octet representation of $q-A_2$ where the decomposition of the coproduct 8×8 is not multiplicity free. In terms of the fusion procedure [12] some solutions beyond the minimal representations can be obtained.

In this paper, we are going to compute the solutions to the Yang–Baxter equation for the non-minimal and important representations, the spinor ones, of the quantum B_l $(q-B_l)$. At first, we will find the explicit form of the generator e_0 corresponding to the lowest negative root. Secondly, in terms of the systematic method based on Jimbo's theorem [7] we will compute the solutions $\check{R}_q(x)$ expressed as a sum of the quantum projectors which are the product of two quantum Clebsch–Gordan coefficients. Thirdly,

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the rational solutions can be calculated through an appropriate limit process. At last, by removing the spectrum parameter x, we can obtain the representations of the braid group, and the link polynomials by a standard method [3].

From a mathematical viewpoint and for analysing physical quantities of the models, it is important to obtain the explicit forms of the quantum projectors. In this paper, we compute, as an example, the explicit forms of the quantum Clebsch-Gordan coefficients, the quantum projectors and the $\check{R}_q(x)$ matrix for the spinor representation of $q-B_3$. Recently, Ge *et al* [13] calculated the $\check{R}_q$ without a spectrum parameter in terms of the generalized Kauffman's [14] state model. The result coincides, except for a misprint in [13], with that obtained from our $\check{R}_q(x)$ by making the spectrum parameter x vanish. In the solvable statistical models the spectrum-dependent solutions are directly related to the Boltzmann weights, and may be more interesting.

The systematic method for constructing the spectrum-dependent solutions to the Yang-Baxter equation based on Jimbo's theorem [7] was introduced in our previous paper [10]. In this paper we use the same notation as in our previous paper.

The plan of this paper is as follows. In section 2, we will find the explicit form $D_q^{N_0}(e_0)$ of the generator e_0 in the spinor representation of $q-B_3$. For the decomposition of the coproduct in two spinor representation spaces, we obtain the quantum Clebsch-Gordan coefficients through a straightforward calculation in section 3. The quantum projectors and the solution to the Yang-Baxter equation for the spinor representation of $q-B_3$ are also listed in section 3. Generalizing the results for $q-B_3$, we obtain the trigonometric solution for the spinor representations of $q-B_l$ in section 4. Through an appropriate limit process the rational solutions are obtained in section 5. Removing the spectrum parameter in the trigonometric solution, we obtain the representations of the braid group and the link polynomials associated with the spinor representations of $q-B_l$ by a standard method in section 6.

2. Generators in the spinor representation of $q-B_3$

The states in the spinor representation N_0 of the quantum B_3 universal enveloping algebra ($q-B_3$) are denoted by the weights

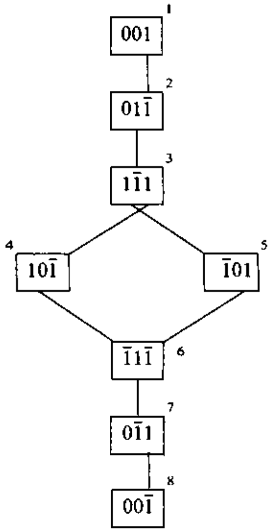
$$m=\sum_{j=1}^{3}(m)_j\lambda_j$$

where λ_j are the fundamental weights, and N_0 is the highest weight in the representation, $N_0=\lambda_3$. For simplicity we enumerate the states in this representation as in table 1, and sometimes use the enumerations to denote the states.

According to the properties [10] of the subalgebras $q-sl(2)$ in $q-B_3$, we can obtain the representation matrices of the generators h_j, e_j and f_j. Hereafter, if no confusion, h_j, e_j and f_j denote the generator operators as well as their representation matrices in the spinor representation, so do those with $j=0$.

$$\begin{aligned}
h_1 &\equiv D_q^{N_0}(h_1)=E_{33}+E_{44}-E_{55}-E_{66}\\
h_2 &\equiv D_q^{N_0}(h_2)=E_{22}-E_{33}+E_{66}-E_{77}\\
h_3 &\equiv D_q^{N_0}(h_3)=E_{11}-E_{22}+E_{33}-E_{44}+E_{55}-E_{66}+E_{77}-E_{88}\\
e_1 &\equiv D_q^{N_0}(e_1)=\tilde{f}_1\equiv\widetilde{D_q^{N_0}(f_1)}=E_{35}+E_{46}\\
e_2 &\equiv D_q^{N_0}(e_1)=\tilde{f}_2\equiv\widetilde{D_q^{N_0}(f_2)}=E_{23}+E_{67}\\
e_3 &\equiv D_q^{N_0}(e_3)=\tilde{f}_3\equiv\widetilde{D_q^{N_0}(f_3)}=E_{12}+E_{34}+E_{56}+E_{78}
\end{aligned}\qquad(1)$$

Table 1. Enumerations of the eight states in the spinor representation of $q-B_3$.



where the tilde denotes the transpose, and E_{ij} is the unit 8×8 matrix

$$(E_{ij})_{kl}=\delta_{ik}\delta_{jl}. \tag{2}$$

In fact, the forms of the generators in the spinor representation of $q-B_3$ are the same as those of B_3.

In Lie algebra B_3, the lowest negative root r_0 is

$$r_0=-r_1-2r_2-2r_3=-\lambda_2 \tag{3}$$

and the corresponding generators are

$$E_0=\tilde{F}_0=\tfrac{1}{2}[[[[f_3,f_2],f_1],f_3],f_2]. \tag{4a}$$

Define

$$\begin{aligned}
&e_0=\tilde{f}_0=E_0=E_{71}+E_{82}\\
&h_0=-h_1-2h_2-h_3\\
&k_0=q^{h_0}=k_1^{-1}k_2^{-2}k_3^{-2}\\
&k_1=q^{h_1}\qquad k_2=q^{h_2}\qquad k_3=q^{1/2h_3}.
\end{aligned} \tag{4b}$$

It is easy to check that those generators satisfy the quantum algebraic relations of $q-B_3$ [8, 10]:

$$\begin{aligned}
&[h_i,h_j]=0\\
&[h_i,e_j]=a_{ij}e_j\qquad [h_i,f_j]=-a_{ij}f_j\\
&[e_i,f_j]=\delta_{ij}\frac{q_j^{2h_j}-q_j^{-2h_j}}{q_j^2-q_j^{-2}}=\delta_{ij}\frac{k_j^2-k_j^{-2}}{q_j^2-q_j^{-2}}\\
&\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_{q_i^2}e_i^{1-a_{ij}-n}e_je_i^n=0\qquad i\neq j\\
&\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_{q_i^2}f_i^{1-a_{ij}-n}f_jf_i^n=0\qquad i\neq j
\end{aligned} \tag{5}$$

where $i, j=0, 1, 2, 3$, a_{ij} is the Cartan matrix, $q_1=q_2=q$, $q_3=q^{1/2}$, and

$$[m]_q=\frac{q^m-q^{-m}}{q-q^{-1}}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q=\frac{[n]_q[n-1]_q\ldots[n-m+1]_q}{[m]_q[m-1]_q\ldots[1]_q}. \tag{6}$$

Usually, we neglect the subscript of $[m]_q$ when the subscript is q. The property, that the generators have the same form for the spinor representation in both B_3 and $q-B_3$, holds for the spinor representations of $q-B_l$, because the eigenvalues of h_j for these representations are only ±1 or 0.

3. Trigonometric solution for $q-B_3$

In B_3 the decomposition of the direct product of two spinor representations are

$$N_0\otimes N_0=N_1\oplus N_2\oplus N_3\oplus N_4 \tag{7}$$

where $N_1=2\lambda_3$, $N_2=\lambda_2$, $N_3=\lambda_1$ and $N_4=0$. The Clebsch-Gordan series for the decomposition of the coproduct Δ in the direct product of two spinor representation spaces of $q-B_3$ is the same as that for B_3 shown in (7).

In calculating the quantum Clebsch-Gordan coefficients we need to know the combinations for the highest weights of N and the representation matrices of the generators f_j in N, where N denotes one of $N_1, N_2, \ldots, N_4$. Denote the states in N by $|N, m\rangle$ and the states in the coproduct by $|m_1\rangle|m_2\rangle$, where m_1 and m_2 are given by their enumerations. From the conditions for the highest weights

$$e_j|N, N\rangle=0$$

we have

$$|N_1, 002\rangle=|1\rangle|1\rangle$$

$$|N_2, 010\rangle=\frac{1}{[2]^{1/2}}\{q^{-1/2}|1\rangle|2\rangle-q^{1/2}|2\rangle|1\rangle\}$$

$$|N_3, 100\rangle=\left(\frac{[3]}{[6][2]}\right)^{1/2}\{q^{-2}|1\rangle|4\rangle-q^{-1}|2\rangle|3\rangle+q|3\rangle|2\rangle-q^2|4\rangle|1\rangle\}$$

$$\begin{aligned}|N_4, 000\rangle=\left(\frac{[5][3]}{[10][6][2]}\right)^{1/2}&\{q^{-9/2}|1\rangle|8\rangle-q^{-7/2}|2\rangle|7\rangle+q^{-3/2}|3\rangle|6\rangle-q^{-1/2}|4\rangle|5\rangle\\&-q^{1/2}|5\rangle|4\rangle+q^{3/2}|6\rangle|3\rangle-q^{7/2}|7\rangle|2\rangle+q^{9/2}|8\rangle|1\rangle\}.\end{aligned} \tag{8}$$

The representation matrices of the generators f_j in the representation N can be calculated by making use of the properties of the subalgebras $q-sl(2)$. The skill used in [10] is useful when the weights are multiple. For example, the weight (100) is doubly multiple in N_1. We assume

$$f_2|N_1, 02\bar{2}\rangle=a|N_1,(100)_1\rangle+b|N_1,(100)_2\rangle \tag{9a}$$

and we know from the properties of subalgebras $q-sl(2)$

$$f_2|N_1, 010\rangle = |N_1, 1\bar{1}2\rangle$$

$$e_3|N_1, 02\bar{2}\rangle = [2]^{1/2}|N_1, 010\rangle$$

$$f_3|N_1, 1\bar{1}2\rangle = [2]^{1/2}|N_1, (100)_1\rangle$$

$$e_1|N_1, (100)_1\rangle = [2]^{1/2}|N_1, 1\bar{1}2\rangle$$

$$e_3|N_1, (100)_2\rangle = 0$$

where $\bar{1}$, for example, means -1. Now, we can calculate the coefficients a and b

$$f_3e_3f_2|N_1, 02\bar{2}\rangle = a[2]|N_1, (100)_1\rangle = f_3f_2e_3|N_1, 02\bar{2}\rangle = [2]|N_1, (100)_1\rangle.$$

Therefore, we obtain

$$a = 1 \qquad b = \left\{\frac{[4]}{[2]} - a^2\right\}^{1/2} = \left(\frac{[6]}{[3][2]}\right)^{1/2}. \tag{9b}$$

It can be proved by calculation that the quantum Clebsch–Gordan coefficients for the decomposition of the coproduct in two spinor representations are invariant in the Weyl reflection, so the $q-CG$ matrix is a block matrix composed by four types of the submatrices: eight 1×1 submatrices, twelve 2×2 submatrices, six 4×4 submatrices and one 8×8 submatrix. The quantum projectors and the solution $\check{R}_q(x)$ also have the same submatrix structure.

The first equation of (8) determines the 1×1 submatrix of $q-CG$:

$$(C_q)_{11N_2(002)} = 1 \qquad (C_q)_{11N_2m} = (C_q)_{11N_3m} = (C_q)_{11N_4m} = 0. \tag{10}$$

The Weyl reflections give the rows of all the 1×1 submatrices which are (m_1, m_1) with $m_1 = 1, 2, \ldots, 8$.

The rows of 2×2 submatrices are (m_1, m_2) and (m_2, m_1) where

$$(m_1, m_2) = (1,2), (1,3), (1,5), (2,4), (2,6), (3,4), (3,7),$$
$$(4,8), (5,6), (5,7), (6,8) \text{ and } (7,8). \tag{11a}$$

The rows of 4×4 submatrices are (m_1, m_2), (m_3, m_4), (m_4, m_3) and (m_2, m_1) where

$$(m_1, m_2, m_3, m_4) = (1,4,2,3), (1,6,2,5)\ (1,7,3,5),$$
$$(2,8,4,6), (3,8,4,7) \text{ and } (5,8,6,7). \tag{11b}$$

The rows of the 8×8 submatrix are

$$(1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1). \tag{11c}$$

The $q-CG$ coefficients are listed in table 2, where the last lines in the tables denote the common factors for the corresponding columns. For example, from table 2(b) we read

$$(C_q)_{23N_2(100)} = \frac{1}{[2]}q^{-1}.$$

It can be shown from table 2 that the quantum Clebsch–Gordan coefficients have the following symmetries:

$$(C_q)_{m_1 m_2 Nm} = \xi_N (C_{q^{-1}})_{m_2 m_1 Nm} \tag{12a}$$

where

$$\xi_{N_1} = -\xi_{N_2} = -\xi_{N_3} = \xi_{N_4} = 1. \tag{12b}$$

Since the decomposition of the coproduct in two spin representations are multiplicity free, the spectrum-dependent solution to the Yang–Baxter equation can be expressed as a sum of the quantum projectors due to the Schur theorem [10]

$$\check{R}_q(x) = \sum_{\mu} \Lambda_{N_\mu}(x, q) \mathscr{P}_{N_\mu} \tag{13}$$

$$\mathscr{P}_{N_\mu} = (C_q)_{N_\mu} (\tilde{C}_q)_{N_\mu} \tag{14}$$

where $\Lambda_{N_\mu}(x, q)$ can be calculated according to Jimbo's theorem [7, 10]:

$$\{xX(q)_{N'm',Nm} + Y(q)_{N'm',Nm}\} \Lambda_N(x, q) = \Lambda_{N'}(x, q) \{X(q)_{N'm',Nm} + xY(q)_{N'm',Nm}\} \tag{15a}$$

$$\begin{aligned} X(q)_{N'm',Nm} &= \sum_{n_1 n_1 n_1' n_2'} (C_q)_{n_1' n_2' N'm'} D_q^{N_0}(k_0)_{n_1' n_1} D_q^{N_0}(e_0)_{n_2' n_2} (C_q)_{n_1 n_2 Nm} \\ Y(q)_{N'm',Nm} &= \sum_{n_1 n_2 n_1' n_2'} (C_q)_{n_1' n_2' N'm'} D_q^{N_0}(e_0)_{n_1' n_1} D_q^{N_0}(k_0^{-1})_{n_2' n_2} (C_q)_{n_1 n_2 Nm} \end{aligned} \tag{15b}$$

where $m' = m + r_0$, and $D_q^{N_0}(k_0)$ is diagonal. In fact, we only need to consider the equations (15) with $m = N$ [10]. Through a straightforward calculation, we obtain the

Table 2. The quantum Clebsch–Gordan coefficients for the coproduct in the direct product of two spinor representation spaces for $q - B_3$. (*a*) 2×2 submatrix, (*b*) 4×4 submatrix, (*c*) 8×8 submatrix.

(*a*)	N_1	N_2
1, 2	$q^{1/2}$	$q^{-1/2}$
2, 1	$q^{-1/2}$	$-q^{1/2}$
Factor	$\frac{1}{[2]^{1/2}}$	$\frac{1}{[2]^{1/2}}$

(*b*)		N_1	N_2	N_3
1, 4	q	$-q$	1	q^{-2}
2, 3	1	q^2	q^{-1}	$-q^{-1}$
3, 2	1	q^{-2}	$-q$	q
4, 1	q^{-1}	$-q^{-1}$	-1	$-q^2$
Factor	$\frac{1}{[2]}$	$\left(\frac{[3]}{[6][2]}\right)^{1/2}$	$\frac{1}{[2]}$	$\left(\frac{[3]}{[6][2]}\right)^{1/2}$

(*c*)		N_1			N_2		N_3	N_4
1, 8	$q^{3/2}$	$-q^{3/2}$	$-q^{3/2}$	0	$-q^{-3/2} - q^{5/2}$	$q^{1/2}$	$q^{-3/2}$	$q^{-9/2}$
2, 7	$q^{1/2}$	$-q^{1/2}$	$q^{5/2}$	0	$q^{-1/2} + q^{7/2}$	$q^{-1/2}$	$q^{-5/2}$	$-q^{-7/2}$
3, 6	$q^{1/2}$	$q^{5/2}$	$q^{-3/2}$	$q^{1/2}$	$q^{-9/2} - q^{3/2}$	$q^{-1/2}$	$-q^{-1/2}$	$q^{-3/2}$
4, 5	$q^{-1/2}$	$q^{3/2}$	$-q^{-1/2}$	$-q^{3/2}$	$-q^{-7/2} + q^{5/2}$	$q^{-3/2}$	$-q^{-3/2}$	$-q^{-1/2}$
5, 4	$q^{1/2}$	$q^{-3/2}$	$-q^{1/2}$	$q^{-3/2}$	$-q^{-5/2} + q^{7/2}$	$-q^{3/2}$	$q^{3/2}$	$-q^{1/2}$
6, 3	$q^{-1/2}$	$q^{-5/2}$	$q^{3/2}$	$-q^{-1/2}$	$q^{-3/2} - q^{9/2}$	$-q^{1/2}$	$q^{1/2}$	$q^{3/2}$
7, 2	$q^{-1/2}$	$-q^{-1/2}$	$q^{-5/2}$	0	$-q^{-7/2} - q^{1/2}$	$-q^{1/2}$	$-q^{5/2}$	$-q^{7/2}$
8, 1	$q^{-3/2}$	$-q^{-3/2}$	$-q^{-3/2}$	0	$q^{-5/2} + q^{3/2}$	$-q^{-1/2}$	$-q^{3/2}$	$q^{9/2}$
Factor	$\frac{1}{[2]^{3/2}}$	$\frac{1}{[2]}\left(\frac{[3]}{[6]}\right)^{1/2}$	$\frac{1}{[2]}\left(\frac{[3]}{[6]}\right)^{1/2}$	$\frac{1}{[4]^{1/2}}$	$\left(\frac{[5]}{[10][4][2]}\right)^{1/2}$	$\frac{1}{[2]^{3/2}}$	$\frac{1}{[2]}\left(\frac{[3]}{[6]}\right)^{1/2}$	$\left(\frac{[5][3]}{[10][6][2]}\right)^{1/2}$

following non-trivial constaints:

$$\begin{aligned}&X(q)_{N(N+r_0),NN}=Y(q)_{N(N+r_0),NN}\\&-q^{C_2(N)-C_2(N')}X(q)_{N'(N+r_0),NN}=Y(q)_{N'(N+r_0),NN}\end{aligned} \tag{16}$$

where the pairs (N, N') or (N', N) are (N_1, N_2), (N_1, N_3) and (N_2, N_4), and $C_2(N)$ denotes the Casimir operators:

$$C_2(N_1)=12 \qquad C_2(N_2)=10 \qquad C_2(N_3)=6 \qquad C_2(N_4)=0. \tag{17}$$

Choosing

$$\Lambda_{N_1}(x,q)=(1-xq^2)(1-xq^6)(1-xq^{10}) \tag{18a}$$

we have

$$\begin{aligned}&\Lambda_{N_2}(x,q)=(x-q^2)(1-xq^6)(1-xq^{10})\\&\Lambda_{N_3}(x,q)=(1-xq^2)(x-q^6)(1-xq^{10})\\&\Lambda_{N_4}(x,q)=(x-q^2)(1-xq^6)(x-q^{10}).\end{aligned} \tag{18b}$$

If we define

$$\check{R}_q\equiv\check{R}_q(0)=\mathcal{P}_{N_1}-q^2\mathcal{P}_{N_2}-q^6\mathcal{P}_{N_3}+q^{12}\mathcal{P}_{N_4} \tag{19a}$$

and

$$\check{R}_q^{-1}=\lim_{x\to\infty}-q^{-18}x^{-3}\check{R}_q(x)=\mathcal{P}_{N_1}-q^{-2}\mathcal{P}_{N_2}-q^{-6}\mathcal{P}_{N_3}+q^{-12}\mathcal{P}_{N_4} \tag{19b}$$

then both $\check{R}_q$ and $\check{R}_q^{-1}$ are symmetric matrices and satisfy

$$(\check{R}_{q^{-1}})_{m_1m_2,m_1'm_2'}=(\check{R}_q^{-1})_{m_2m_1,m_2'm_1'}. \tag{20}$$

It is easy to prove that $\check{R}_q(x)$ is also a symmetric matrix and satisfies

$$\check{R}_q(x)_{m_1m_2,m_1'm_2'}=-q^{18}x^3\check{R}_{q^{-1}}(x^{-1})_{m_2m_1,m_2'm_1'}. \tag{21}$$

Therefore, we obtain

$$\check{R}_q(x)=\check{R}_q+xS_q-x^2q^{18}S_q'-x^3q^{18}\check{R}_q^{-1}$$

$$(S'_{q^{-1}})_{m_1m_2,m_1'm_2'}=(S_q)_{m_2m_1,m_2'm_1'}$$

$$\begin{aligned}S_q&=-(q^2+q^6+q^{10})\mathcal{P}_{N_1}+(1+q^8+q^{12})\mathcal{P}_{N_2}\\&\quad+(1+q^8+q^{16})\mathcal{P}_{N_3}-(q^2+q^{10}+q^{18})\mathcal{P}_{N_4}\\&=(1-q^2-q^6+q^8)\mathbb{1}-(1+q^{10})\check{R}_q-q^8\check{R}_q^{-1}+q^7(1-q^2)^2\frac{[10][6][2]}{[5]}\mathcal{P}_{N_4}.\end{aligned}$$

The quantum projectors $\mathcal{P}_N$ and $\check{R}_q$, S_q all are the same type block matrices as $q-CG$ matrix. The explicit forms of the submatrices are as follows:

(i) Eight 1×1 submatrices:

$$\mathcal{P}_{N_1}=1 \qquad \mathcal{P}_{N_2}=\mathcal{P}_{N_3}=\mathcal{P}_{N_4}=0$$

$$\check{R}_q=1 \qquad S_q=-q^2-q^6-q^{10}$$

(ii) Twelve 2×2 submatrices:

$$\mathscr{P}_{N_1}=\frac{1}{[2]}\begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix} \qquad \mathscr{P}_{N_1}=\frac{1}{[2]}\begin{pmatrix} q^{-1} & -1 \\ -1 & q \end{pmatrix} \qquad \mathscr{P}_{N_3}=\mathscr{P}_{N_4}=0$$

$$\check{R}_q=\begin{pmatrix} 0 & q \\ q & 1-q^2 \end{pmatrix} \qquad S_q=\begin{pmatrix} 1-q^2 & -q-q^7-q^{11} \\ -q-q^7-q^{11} & -q^6+q^8-q^{10}+q^{12} \end{pmatrix}$$

(iii) Six 4×4 submatrices:

$$\mathscr{P}_{N_1}=\frac{[3]}{[6][2]}\begin{pmatrix} 1+q^4 & q^{-1}-q & -q+q^3 & q^{-2}+q^2 \\ q^{-1}-q & q^{-2}-1+q^2+q^4 & q^{-2}+q^2 & q^{-3}-q^{-1} \\ -q+q^3 & q^{-2}+q^2 & q^{-4}+q^{-2}-1+q^2 & -q^{-1}+q \\ q^{-2}+q^2 & q^{-3}-q^{-1} & -q^{-1}+q & q^{-4}+1 \end{pmatrix}$$

$$\mathscr{P}_{N_2}=\frac{1}{[2]^2}\begin{pmatrix} 1 & q^{-1} & -q & -1 \\ q^{-1} & q^{-2} & -1 & -q^{-1} \\ -q & -1 & q^2 & q \\ -1 & -q^{-1} & q & 1 \end{pmatrix}$$

$$\mathscr{P}_{N_3}=\frac{[3]}{[6][2]}\begin{pmatrix} q^{-4} & -q^{-3} & q^{-1} & -1 \\ -q^{-3} & q^{-2} & -1 & q \\ q^{-1} & -1 & q^2 & -q^3 \\ -1 & q & -q^3 & q^4 \end{pmatrix}$$

$$\mathscr{P}_{N_4}=0$$

$$\check{R}_q=\begin{pmatrix} 0 & 0 & 0 & q^2 \\ 0 & 0 & q^2 & q-q^3 \\ 0 & q^2 & 1-q^4 & -q^3+q^5 \\ q^2 & q-q^3 & -q^3+q^5 & (1-q^2)(1+q^4) \end{pmatrix}$$

$$S_q=\begin{pmatrix} 1-q^4 & -q^3+q^5 & q^5-q^7 & -q^2-q^6-q^{12} \\ -q^3+q^5 & 1-q^2+q^4-q^6 & -q^2-q^6-q^{12} & -q+q^3-q^{11}+q^{13} \\ q^5-q^7 & -q^2-q^6-q^{12} & -q^2+q^4-q^6+q^8-q^{10}+q^{14} & q^3-q^5+q^{13}-q^{15} \\ -q^2-q^6-q^{12} & -q+q^3-q^{11}+q^{13} & q^3-q^5+q^{13}-q^{15} & -q^4+q^8-q^{10}+q^{12}-q^{14}+q^{16} \end{pmatrix}.$$

(iv) One 8×8 submatrix. We introduce some symbols to simplify the expressions:

$$a=q^{-1}-q \qquad b=q^{-2}+q^2=[4]/[2] \qquad c=q^{-5}+q^5=[10]/[5]$$

$$d=q^{-4}+1+q^4=[6]/[2] \qquad u=q^{-3}+q-q^3 \qquad v=-q^{-3}+q^{-1}+q^3.$$

$$\mathscr{P}_{N_1}=\frac{[3]}{[6][2]^2}\begin{pmatrix} q^3[3] & 1 & -q^2 & q^{-1} & q^4 & -q & q^3 & [3] \\ 1 & q^2[4]-q & q^{-1} & a[2]-1 & -q & q^3[2]-1 & [3] & q^{-3} \\ -q^2 & q^{-1} & q([5]-2) & q^3[2]-1 & q^3 & [3] & q^{-3}[2]-1 & -q^{-1} \\ q^{-1} & a[2]-1 & q^3[2]-1 & [4]-q^{-1} & [3] & q^{-3} & -q^{-1} & q^{-4} \\ q^4 & -q & q^3 & [3] & [4]-q & q^{-3}[2]-1 & 2q^2-[3] & q \\ -q & q^3[2]-1 & [3] & q^{-3} & q^{-3}[2]-1 & q^{-1}([5]-2) & q & -q^{-2} \\ q^3 & [3] & q^{-3}[2]-1 & -q^{-1} & 2q^2-[3] & q & q^{-2}([4]-q) & 1 \\ [3] & q^{-3} & -q^{-1} & q^{-4} & q & -q^{-2} & 1 & q^{-3}[3] \end{pmatrix}$$

$$\mathscr{P}_{N_1}=\frac{[5]}{[10][2]^2}\begin{pmatrix} qd & -q^{-1}v & qv & q^{-2}u & -q^3v & -u & q^2u & -d \\ -q^{-1}v & c+v & q^{-2}u & q^{-1}(c-u) & -u & -q(c-u) & -d & q^{-2}v \\ qv & q^{-2}u & q^{-1}[2]c-u & -q(c-u) & q^2u & -d & -q^{-1}(c-v) & -v \\ q^{-2}u & q^{-1}(c-u) & -q(c-u) & q^6[2]+q^{-4}u & -d & q^{-2}v & -v & -q^{-3}u \\ -q^3v & -u & q^2u & -d & q^{-6}[2]+q^4v & -q^{-1}(c-v) & q(c-v) & q^2v \\ -u & -q(c-u) & -d & q^{-2}v & -q^{-1}(c-v) & q[2]c-v & q^2v & q^{-1}u \\ q^2u & -d & -q^{-1}(c-v) & -v & q(c-v) & q^2v & c+u & -qu \\ -d & q^{-2}v & -v & -q^{-3}u & q^2v & q^{-1}u & -qu & q^{-1}d \end{pmatrix}.$$

$$\mathscr{P}_{N_3}=\frac{[3]}{[6][2]^2}\begin{pmatrix} q^{-3} & q^{-4} & -q^{-2} & -q^{-3} & 1 & q^{-1} & -q & -1 \\ q^{-4} & q^{-5} & -q^{-3} & -q^{-4} & q^{-1} & q^{-2} & -1 & -q^{-1} \\ -q^{-2} & -q^{-3} & q^{-1} & q^{-2} & -q & -1 & q^2 & q \\ -q^{-3} & -q^{-4} & q^{-2} & q^{-3} & -1 & -q^{-1} & q & 1 \\ 1 & q^{-1} & -q & -1 & q^3 & q^2 & -q^4 & -q^3 \\ q^{-1} & q^{-2} & -1 & -q^{-1} & q^2 & q & -q^3 & -q^2 \\ -q & -1 & q^2 & q & -q^4 & -q^3 & q^5 & q^4 \\ -1 & -q^{-1} & q & 1 & -q^3 & -q^2 & q^4 & q^3 \end{pmatrix},$$

$$\mathscr{P}_{N_4}=\frac{[5][3]}{[10][6][2]}\begin{pmatrix} q^{-9} & -q^{-8} & q^{-6} & -q^{-5} & -q^{-4} & q^{-3} & -q^{-1} & 1 \\ -q^{-8} & q^{-7} & -q^{-5} & q^{-4} & q^{-3} & -q^{-2} & 1 & -q \\ q^{-6} & -q^{-5} & q^{-3} & -q^{-2} & -q^{-1} & 1 & -q^2 & q^3 \\ -q^{-5} & q^{-4} & -q^{-2} & q^{-1} & 1 & -q & q^3 & -q^4 \\ -q^{-4} & q^{-3} & -q^{-1} & 1 & q & -q^2 & q^4 & -q^5 \\ q^{-3} & -q^{-2} & 1 & -q & -q^2 & q^3 & -q^5 & q^6 \\ -q^{-1} & 1 & -q^2 & q^3 & q^4 & -q^5 & q^7 & -q^8 \\ 1 & -q & q^3 & -q^4 & -q^5 & q^6 & -q^8 & q^9 \end{pmatrix}$$

$$\check{R}_q=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^3 & q^3a \\ 0 & 0 & 0 & 0 & 0 & q^3 & q^3[2]a & -q^5a \\ 0 & 0 & 0 & 0 & q^3 & q^3a & -q^5a & q^4ab \\ 0 & 0 & 0 & q^3 & 0 & q^3[2]a & -q^5[2]a & q^7a \\ 0 & 0 & q^3 & q^3a & q^3[2]a & q^3[2]a^2 & q^7a & -q^6ab \\ 0 & q^3 & q^3[2]a & -q^5a & -q^5[2]a & q^7a & q^5[4]a^2 & q^8ab \\ q^3 & q^3a & -q^5a & q^4ab & q^7a & -q^6ab & q^8ab & q^6[3]a^2b \end{pmatrix}$$

$$S_q=\begin{pmatrix} 0 & q^3[2]a & -q^5[2]a & q^7a & q^7[2]a & -q^9a & q^{11}a & -q^7d \\ q^3[2]a & q^3[2]a^2 & q^7a & -q^6ab & -q^9a & q^8ab & -q^7d & -q^5ab \\ -q^5[2]a & q^7a & q^5[4]a^2 & q^8ab & q^{11}a & -q^7d & -q^6a(qc+b) & q^7ab \\ q^7a & -q^6ab & q^8ab & q^6[3]a^2b & -q^7d & -q^5ab & q^7ab & -q^7a(c+qb) \\ q^7[2]a & -q^9a & q^{11}a & -q^7d & q^7[6]a^2 & -q^6a(qc+b) & q^8a(qc+b) & -q^9ab \\ -q^9a & q^8ab & -q^7d & -q^5ab & -q^6a(qc+b) & q^8a^2b & -q^9ab & q^9a(c+qb) \\ q^{11}a & -q^7d & -q^6a(qc+b) & q^7ab & q^8a(qc+b) & -q^9ab & -q^{10}a^2b & -q^{11}a(c+qb) \\ -q^7d & -q^5ab & q^7ab & -q^7a(c+qb) & -q^9ab & q^9a(c+qb) & -q^{11}a(c+qb) & -q^{11}[6]a^2 \end{pmatrix}.$$

Ge *et al* [13] calculated the solution $\check{R}_q$ for the spinor representation of $q-B_3$ by the generalized Kauffman's state models. In their notation, their t is equal to our $q^{-1/2}$, and their $t^{-3}S$ is equal to our $\check{R}_q$. Two results from two different methods are coincident except for the 1×1 submatrices. The difference comes from an obvious misprint in [13].

4. Trigonometric solutions for $q-B_l$

For B_l the decomposition of the direct product of two spinor representations $N_0=\lambda_l$ are

$$N_0\otimes N_0=N_1\oplus N_2\oplus\ldots\oplus N_{l+1}$$
$$N_1=2\lambda_l \qquad N_{l+1}=0 \tag{22}$$
$$N_n=\lambda_{l-n+1} \qquad 2\leqslant n\leqslant l.$$

The Casimir operators are

$$C_2(N_n)=l(l+1)-n(n-1) \qquad 1\leqslant n\leqslant l+1. \tag{23}$$

The representation matrices of the generators in the spinor representations of $q-B_l$ are the same as those of B_l. Corresponding to the lowest negative root r_0

$$r_0=-r_1-2\sum_{j=1}^{l} r_j$$

the generators are

$$h_0=-h_1-h_l-2\sum_{j=2}^{l-1} h_j \tag{24a}$$

$$e_0=\tfrac{1}{2}[\ldots[f_l,f_{l-1}],f_{l-2}]\ldots,f_1],f_l],f_{l-1}],\ldots,f_2]. \tag{24b}$$

Now the key for calculating the quantum Clebsch-Gordan coefficients and the quantum projectors is to determine the explicit expansions of the highest weight states which are easy to be obtained from the conditions that the highest weight states are annihilated by the raising operator $\Delta(e_j)$, $0<j\leqslant l$. Clearly, we have

$$|N_1,2\lambda_l\rangle=|\lambda_l\rangle|\lambda_l\rangle$$

$$|N_2,\lambda_{l-1}\rangle=[2]^{-1/2}\{q^{-1/2}|\lambda_l\rangle|\lambda_{l-1}-\lambda_l\rangle-q^{1/2}|\lambda_{l-1}-\lambda_l\rangle|\lambda_l\rangle\}$$

$$\begin{aligned}|N_3,\lambda_{l-2}\rangle=\left(\frac{[3]}{[6][2]}\right)^{1/2}&\{q^{-2}|\lambda_l\rangle|\lambda_{l-2}-\lambda_l\rangle-q^{-1}|\lambda_{l-1}-\lambda_l\rangle|\lambda_{l-2}-\lambda_{l-1}+\lambda_l\rangle\\&+q|\lambda_{l-2}-\lambda_{l-1}+\lambda_l\rangle|\lambda_{l-1}-\lambda_l\rangle-q^2|\lambda_{l-2}-\lambda_l\rangle|\lambda_l\rangle\}\end{aligned}$$

$$\begin{aligned}|N_4,\lambda_{l-3}\rangle=\left(\frac{[5][3]}{[10][6][2]}\right)^{1/2}&\{q^{-9/2}|\lambda_l\rangle|\lambda_{l-3}-\lambda_l\rangle-q^{-7/2}|\lambda_{l-1}-\lambda_l\rangle|\lambda_{l-3}-\lambda_{l-1}+\lambda_l\rangle\\&+q^{-3/2}|\lambda_{l-2}-\lambda_{l-1}+\lambda_l\rangle|\lambda_{l-3}-\lambda_{l-2}+\lambda_{l-1}-\lambda_l\rangle\\&-q^{-1/2}|\lambda_{l-2}-\lambda_l\rangle|\lambda_{l-3}-\lambda_{l-2}+\lambda_l\rangle-q^{1/2}|\lambda_{l-3}-\lambda_{l-2}+\lambda_l\rangle|\lambda_{l-2}-\lambda_l\rangle\\&+q^{3/2}|\lambda_{l-3}-\lambda_{l-2}+\lambda_{l-1}-\lambda_l\rangle|\lambda_{l-2}-\lambda_{l-1}+\lambda_l\rangle\\&-q^{7/2}|\lambda_{l-3}-\lambda_{l-1}+\lambda_l\rangle|\lambda_{l-1}-\lambda_l\rangle+q^{9/2}|\lambda_{l-3}-\lambda_l\rangle|\lambda_l\rangle\}.\end{aligned}$$

The rule is very clear. For N_n, the highest weight is $N_n=\lambda_{l-n+1}$. The first term in the expansion of the highest weight state is $cq^{-\alpha}|\lambda_l\rangle|\lambda_{l-n+1}-\lambda_l\rangle$ where c is the normalization factor, and α will be determined later. The difference between λ_l and $\lambda_{l-n+1}-\lambda_l$ can be expressed by simple roots of B_l:

$$\lambda_l-(\lambda_{l-n+1}-\lambda_l)=\sum_{j=1}^{n-1}(n-j)r_{l-j+1}.$$

There is a recursive way of obtaining the terms in the expansion from the preceding term. Assume that we have a term $\varepsilon c q^{\beta}|m_1\rangle|m_2\rangle$ with $\varepsilon=1$ or -1, and $m_1+m_2=\lambda_{l-n+1}$ in the expansions. m_1 is the algebraic sum of some fundamental weight λ_j. If m_1 contains a positive λ_j, $j>l-n+1$, there is a term $-\varepsilon c q^{\beta+s}|m_1-r_j\rangle|m_2+r_j\rangle$ in the expansion where $s=1$ if $j=l$, and $s=2$ if $j\neq l$. In this way we can obtain all the terms in the expansion of the highest weight state from the first term. In particular, the last term is $c\xi_n q^{\alpha}|\lambda_{l-n+1}-\lambda_l\rangle|\lambda_l\rangle$ where

$$2\alpha=(n-1)+2\sum_{j=2}^{n-1}(n-j)=(n-1)^2$$

$$\xi_n=(-1)^{\sum_{j=1}^{n-1}(n-j)}=(-1)^{n(n-1)/2}.$$

The normalization factor is proved to be

$$c=\prod_{j=0}^{n-2}\left(\frac{[1+2j]}{[2+4j]}\right)^{1/2}.$$

It is straightforward to calculate the expansions of all the states in the representation N_n by using the lowering operator $\Delta(f_j)$, $0<j\leqslant l$, on the highest weight state. The coefficients in the expansions are just the quantum Clebsch-Gordan coefficients. Note that ξ_n describes the symmetry of the Clebsch-Gordan coefficients (see (12)). The quantum projectors are the product of two quantum Clebsch-Gordan matrices.

Our next task is to show how many pairs (N, N') or (N', N) occur in (16). It is easy to check that $\lambda_j+r_0=\lambda_j-\lambda_2$ is a Weyl reflection of λ_{j-2} $(j>2)$. Therefore, the only possible pairs are (N_n, N_{n+1}) or (N_n, N_{n+2}). When $k_0\otimes e_0$ and $e_0\otimes k_0^{-1}$ act on the highest weight state of N_n, $n<l$, they do not change the relative ratio of the neighbouring terms. From the expansions of the highest weight states and the next highest weight states, we can obtain (16) with the only possible pairs

$$(N_1, N_2)\qquad\text{and}\qquad(N_n, N_{n+2})\qquad 1\leqslant n\leqslant l-1.$$

This conclusion coincides with a theorem [17] that only those representations with the different ξ can occur in the pair.

From these properties we obtain the trigonometric solutions to the Yang-Baxter equation for the spinor representations of $q-B_l$ as follows

$$\begin{aligned}\check{R}_q(x)=&(1-xq^2)(1-xq^6)\dots(1-xq^{4l-2})\mathcal{P}_{2\lambda_l}\\&+\sum_{n=1}^{n_1}\prod_{i=1}^{n}(x-q^{8i-6})\prod_{j=n+1}^{n_1}(1-xq^{8j-6})\prod_{k=1}^{n_2}(1-xq^{8k-2})\mathcal{P}_{\lambda_{l-2n+1}}\\&+\sum_{n=1}^{n_2}\prod_{i=1}^{n_1}(1-xq^{8i-6})\prod_{j=1}^{n}(x-q^{8j-2})\prod_{k=n+1}^{n_2}(1-xq^{8k-2})\mathcal{P}_{\lambda_{l-2n}}\end{aligned}\qquad(25)$$

where $\lambda_0=0$ and

$$\begin{aligned}&n_1=n_2=l/2 &&\text{when } l \text{ is even}\\&n_1=n_2+1=(l+1)/2 &&\text{when } l \text{ is odd.}\end{aligned}\qquad(26)$$

Removing the spectrum parameter x, we have

$$\check{R}_q \equiv \check{R}_q(0) = \mathscr{P}_{2\lambda_l} + \sum_{n=1}^{l} (-1)^{n(n+1)/2} q^{n(n+1)} \mathscr{P}_{\lambda_{l-n}}$$

$$\check{R}_q^{-1} = \lim_{X\to\infty} (-1)^l q^{-2l^2} x^{-l} \check{R}_q(x)$$

$$= \mathscr{P}_{2\lambda_l} + \sum_{n=1}^{l} (-1)^{n(n+1)/2} q^{-n(n+1)} \mathscr{P}_{\lambda_{l-n}} \tag{27}$$

where the quantum projectors $\mathscr{P}_N$ are the product of the quantum Clebsch-Gordan coefficients

$$(\mathscr{P}_N)_{m_1m_2,m_1'm_2'} = \sum_m (C_q)_{m_1m_2Nm}(C_q)_{m_1'm_2'Nm} \tag{28}$$

where the summation is over the multiple weights $m = m_1 + m_2 = m_1' + m_2'$. If m is a simple weight, $\mathscr{P}_N$ only contains one term. The explicit forms of $\mathscr{P}_N$ can be calculated in a straightforward but tedious way.

5. The rational solutions for $q - B_l$

The rational solutions $R(u, \eta)$ can be obtained from the trigonometric ones through the following limit process [7, 15]:

$$R(u,\eta) = P\check{R}(u,\eta) = \lim_{q\to 1} \frac{P\check{R}_q(q^{2u/\eta})}{(1-q^{2u/\eta})^l}$$

$$= (1+\eta/u)(1+3\eta/u)\dots(1+(2l-1)\eta/u)P_{2\lambda_l}$$

$$+ \sum_{n=1}^{n_1} \prod_{i=1}^{n} (1-(4i-3)\eta/u) \prod_{j=n+1}^{n_1} (1+(4j-3)\eta/u)$$

$$\times \prod_{k=1}^{n_2} (1+(4k-1)\eta/u) P_{\lambda_{l-2n+1}}$$

$$+ \sum_{n=1}^{n_2} \prod_{i=1}^{n_1} (1+(4i-3)\eta/u) \prod_{j=1}^{n} (1-(4j-1)\eta/u)$$

$$\times \prod_{k=n+1}^{n_2} (1+(4k-1)\eta/u) P_{\lambda_{l-2n}} \tag{29}$$

where n_1 and n_2 are given in (26), P denotes the transposition and P_N are the projectors for Lie algebra B_l

$$P_N = \mathscr{P}_N|_{q=1}. \tag{30}$$

For $q - B_3$, we have

$$R(u,\eta) = (1+\eta/u)(1+3\eta/u)(1+5\eta/u)P_{2\lambda_3} + (1-\eta/u)(1+3\eta/u)(1+5\eta/u)P_{\lambda_2}$$
$$+ (1+\eta/u)(1-3\eta/u)(1+5\eta/u)P_{\lambda_1} + (1-\eta/u)(1+3\eta/u)(1-5\eta/u)P_0. \tag{31}$$

6. Representations of the braid group and link polynomials

From the Yang–Baxter equation, the solution $\check{R}_q$ without the spectrum parameter satisfies the similar relations to those in the braid group B_n, so it is easy to obtain a representation of B_n from $\check{R}_q$. The representation matrices of the generators b_i of the braid group, $1 \leqslant i \leqslant n-1$, are defined as

$$D(b_i) = \mathbb{1}^{(1)} \otimes \mathbb{1}^{(2)} \otimes \ldots \otimes \mathbb{1}^{(i-1)} \otimes \check{R}_q \otimes \mathbb{1}^{(i+2)} \otimes \ldots \otimes \mathbb{1}^{(n)}. \tag{32}$$

They satisfy

$$\begin{aligned} &D(b_i)D(b_j) = D(b_j)D(b_j) \qquad |i-j| \geqslant 2 \\ &D(b_i)D(b_{i\pm 1})D(b_i) = D(b_{i\pm 1})D(b_i)D(b_{i\pm 1}). \end{aligned} \tag{33}$$

Since the eigenvalues of $\check{R}_q$ are $(-1)^{j(j+1)/2} q^{j(j+1)}$, $0 \leqslant j \leqslant l$, we have the reduction relation as follows

$$\prod_{j=0}^{l} \{D(b_i) - (-1)^{j(j+1)/2} q^{j(j+1)}\} = 0. \tag{34}$$

Any oriented link is equivalent to a closed braid denoted by $L(A, n)$, $A \in B_n$. From the Markov theorem, the equivalent closed braid can be related by a set of Markov moves. Therefore, a link polynomial $\alpha(A, n)$ corresponding to the closed braid $L(A, n)$ should satisfy the conditions

$$\alpha(AB, n) = \alpha(BA, n) \qquad \alpha(Ab_n^{\pm 1}, n+1) = \alpha(A, n). \tag{35}$$

By making use of the standard method [3], we have the link polynomials associated with the spinor representations of $q - B_l$ as follows

$$\alpha(A, n) = (\tau\bar{\tau})^{-(n-1)/2} \left(\frac{\bar{\tau}}{\tau}\right)^{e(A)/2} \mathrm{Tr}\{VD(A, n)\} \tag{36}$$

where $e(A)$ is the exponent sum of generators in A, and $V = v^{\otimes n}$ [15]

$$v_{mm'} = \delta_{mm'}\{(C_q)_{m\bar{m}00}\}^2 = \delta_{mm'} q^{-4\rho(m)}/d_{N_0}(q) \tag{37}$$

where ρ denotes the Weyl operator

$$\rho(m) = (\rho, m) = \tfrac{1}{2}\sum_{ij} (r_i, r_i)(a^{-1})_{ij}(m)_j = \tfrac{1}{2}\sum_{j=1}^{l-1} (m)_j j(2l-j) + \tfrac{1}{4}(m)_l l^2 \tag{38}$$

and the quantum dimension

$$d_{N_0}(q) = \sum_m q^{-4\rho(m)}. \tag{39}$$

It is easy to prove from the symmetries of the $q - CG$ matrix that

$$\begin{aligned} &[(v \otimes v), \check{R}_q] = 0 \\ &\sum_m \{(\mathbb{1} \otimes v)\check{R}_q\}_{m_1 m, m_2 m} = \delta_{m_1 m_2} \tau \\ &\sum_m \{(\mathbb{1} \otimes v)\check{R}_q^{-1}\}_{m_1 m, m_2 m} = \delta_{m_1 m_2} \bar{\tau} \\ &\tau = q^{-l^2}/d_{N_0}(q) \qquad \bar{\tau} = q^{l^2}/d_{N_0}(q). \end{aligned} \tag{40}$$

The Skein relation can be expressed as

$$\alpha\left(A \prod_{j=0}^{l} (b_i - (-1)^{j(j+1)/2} q^{j(j+1)+l^2}) B, n\right) = 0 \tag{41}$$

where A, $B \in B_n$, $1 \leq i \leq n-1$, and $\alpha(A(c_1A_1+c_2B_1)B, n)$ is understood as $c_1\alpha(AA_1B, n)+c_2\alpha(AB_1B, n)$. For a simple loop, the link polynomial is $\alpha(E, 2)$

$$\alpha(E, 2)=(\tau\bar{\tau})^{-1/2}=d_{N_0}(q). \tag{42}$$

For $q-B_3$, we have

$$4\rho(m)=10(m)_1+16(m)_2+9(m)_3$$

$$\alpha(E, 2)=d_{N_0}(q)=\frac{[10][6][2]}{[5][3]}$$

and the Skein relation is

$$\begin{aligned}\alpha(Ab_i^4B, n)&=q^9(1-q^2-q^6+q^{12})\alpha(Ab_i^3B, n)\\&\quad+q^{20}(1+q^4-q^6-q^{10}+q^{12}+q^{16})\alpha(Ab_i^2B, n)\\&\quad+q^{35}(1-q^6-q^{10}+q^{12})\alpha(Ab_iB, n)-q^{56}\alpha(AB, n).\end{aligned} \tag{43}$$

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